The Onset of Different Modes of Instability for Flow between Rotating Cylinders

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The stability of flow between rotating concentric cylinders was quantitatively investigated by G. I. Taylor (*1*). He found that at the critical point the initial Couette flow pattern would break down into pairs of rotationally symmetric vortices (Figure 1).

Taylor's original criteria for the critical angular velocity are quite general in nature and predict the point of neutral stability for all modes of rotation. Since Taylor's initial work, it has been found that other conditions of flow in the gap, such as circumferential pumping, pumping plus rotation, and reverse flow, also give rise to vortex systems. The vortex pattern in each of these systems may vary, but this pattern may be predicted qualitatively through the use of the Rayleigh criterion (*2*). In brief, Rayleigh's criterion states that the curved flow is unstable if the square of the circulation (characterized by v^2r^2 for circular motion) decreases outwards in the gap; conversely, flow is stable if the square of circulation increases outwards.

At sufficiently high speed all types of flow between cylinders break down into chaotic turbulence; the reason for this interesting fact is not known.

CRITERIA FOR NEUTRAL STABILITY

Taylor's original criterion for neutral stability is

$$\frac{\pi^4 \nu^2 (r_1 + r_2)}{2\omega_1^2 d^3 r_1^2 \left(1 - \mu \dfrac{r_2^2}{r_1^2}\right)(1-\mu)} = 0.0571\left(\frac{1+\mu}{1-\mu} - 0.652\frac{d}{r_1}\right) + 0.00056\left(\frac{1+\mu}{1-\mu} - 0.652\frac{d}{r_1}\right)^{-1} \quad (1)$$

In addition to the above equation, several other criteria have been developed which predict the point of neutral stability for different specific types of flow.

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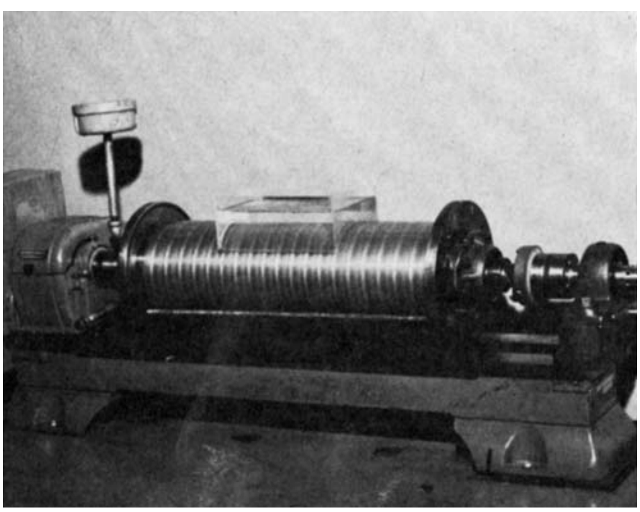

Fig. 1. Horizontal apparatus illustrating steady Taylor vortices.

When only the inner cylinder is rotating, the gap width is potentially unstable, and the vortices which form are contained within approximately square compartments with a wavelength of $2d$. If we consider a small gap $\left(\frac{d}{r_1} \simeq 0\right)$, Equation (1) may be reduced to

$$\frac{\omega_{1c} r_1 d}{\nu}\sqrt{\frac{d}{r_1}} = 41.2 = Ta_c \quad (2)$$

The left-hand side of this equation is called the *Taylor number* and is the product of a gap width Reynolds number and a geometric factor. Kaye and Elgar (*3*) have developed a correction factor, Fg^*, for finite gap widths so

$$^* Fg = \frac{\pi^2}{41.2}\left(1 - \frac{d}{2r_m}\right)^{-1} Z^{-1/2}$$

$$Z = 0.0571\left[1 - 0.652\,\frac{\dfrac{d}{r_m}}{1 - \dfrac{d}{2r_m}}\right] + 0.00056\left[1 - 0.652\,\frac{\dfrac{d}{r_m}}{1 - \dfrac{d}{2r_m}}\right]^{-1}$$

that $Ta_c = 41.2\ Fg$ where Ta_c is the critical Taylor number.

When both cylinders are rotating and $\mu > 1$, the flow is stable at all speeds of rotation until turbulence sets in. For $0 \leq \mu < 1$, Equation (1) or values of $2Ta_c^2$ tabulated by Chandrasekhar (*4*) may be used to predict the point of neutral stability.

For the cylinders rotating in opposite directions μ is negative, and the point of neutral stability may be found from Equation (1) for $-0.5 < \mu < 0$. Chandrasekhar (*4*) and Di Prima (*5*) have also obtained expressions which may be used to predict the point of neutral stability for $\mu < 0$.

In order to have a stability parameter which was not a function of the velocity distribution in the annulus at criticality, Brewster, Grosberg, and Nissan (*6*) have defined

$$P = \frac{\overline{V}\delta}{\nu}\sqrt{\frac{\delta}{r_1}} \quad (3)$$

where $\overline{V}$ is the average velocity in the potentially unstable region δ. The region of instability, δ, may be calculated by considering the radius of the zero velocity streamline as calculated from the Couette flow equation

$$r_o^2 = \frac{(1-\mu)\ (r_2^2 - r_1^2)}{\left[1 - \left(\frac{r_1}{r_2}\right)^2\right]\left[1 - \mu\left(\frac{r_2}{r_1}\right)^2\right]} \quad (4)$$

P, defined as above, is relatively insensitive to the shape of the velocity distribution and is a function of δ/d only. The value of P at neutral stability, P_c, is given by the expressions

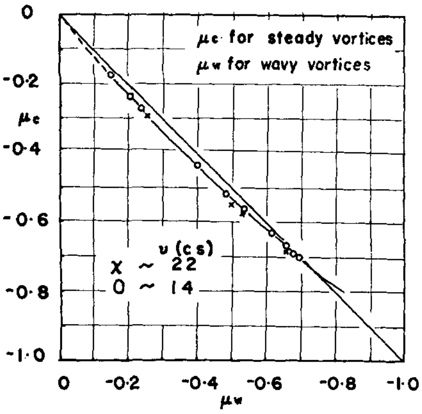

Fig. 2. Speed ratios for steady vortex formation μ_c vs. speed ratios for wavy vortices μ_w for any given rotational speed of the outer cylinder; cylinders in counterrotation.

$$P_c = 12.0;\ 0 < \delta/d < 2/3 \quad (5)$$

$$P_c = 20.7\ (\delta/d)^{5/2} \sqrt{\frac{\dfrac{2\delta}{d} - 1}{\left(\dfrac{2\delta}{d} - 1\right)^2 + 0.01}};\ \frac{2}{3} < \frac{\delta}{d} < 1 \quad (6)$$

NONROTATIONALLY SYMMETRIC FLOW

In addition to the rotationally symmetric (steady) vortices, Taylor (*1*) also observed the formation of nonrotationally symmetric (wavy) vortices. Since 1923, other authors have noted this breakdown of the steady vortices. Schultz-Grunow and Hein (*7*) have published some excellent photographs of the wavy vortices. Coles (*8*) has observed that when only the inner cylinder rotates the amplitude of the steady Taylor vortices increases until a wave motion with four waves in the circumferential direction appears. Di Prima (*9*) has attempted a solution based on linearized disturbance equations for the critical point of nonrotationally symmetric vortices. This solution qualitatively agrees with the observations of Coles in that nonrotationally symmetric disturbances are predicted when Ta is only a little greater than Ta_c.

EXPERIMENTAL APPARATUS

Both a vertical and horizontal apparatus* were used. The vertical apparatus consisted of an inner aluminum cylinder with an O.D. of 2.972 ± 0.001 in. and an acrylic resin outer cylinder with an I.D. of 3.510 ± 0.005 in. The axial length of the cylinders was 30 in. Both cylinders could rotate in either direction. The outer cylinder was surrounded by a plastic box which was filled with water to minimize optical distortion due to the curvature of the cylinders.

The horizontal apparatus consisted of an inner cylinder of stainless steel with an O.D. of 3.470 ± 0.001 in. and an outer cylinder of acrylic resin with an I.D. of 4.225 ± 0.005 in. The axial length of the cylinders was 15 in. Only the inner cylinder was free to rotate.

Visualization of the Flow

Two methods were used to visualize the flow. In one case a mixture of ethyl alcohol and water was added to mineral oil and dispersed into small droplets which would follow the patterns of flow. The second method consisted of mixing aluminum powder with oil.

Ethyl alcohol and water droplets were used exclusively in the vertical equipment. Both methods of indication were used in the horizontal apparatus. The aluminum powder gave better results. In both cases, photographs of observed phenomena were taken.

Procedure

In the vertical equipment, four oils of different viscosity ranges were used. For each oil, the speeds of rotation for both steady and wavy vortex formation were recorded for the rotation of the inner cylinder only.

When both cylinders were rotating in opposite directions, the outer cylinder was set at a fixed revolution per minute and the speed of the inner cylinder gradually increased until the formation of the steady vortex system. The inner cylinder speed was then further increased, the outer cylinder speed being kept constant, until the steady vortex system broke down. The speeds of rotation at the first and second critical points and the frequency of oscillation of the wavy vortices were recorded. A few runs were taken for both cylinders rotating in the same direction. The results are qualitative in nature; the procedure will be given along with these results.

Only one oil was used in the horizontal apparatus. Again, the speeds of rotation for both steady and wavy vortex formation as well as the frequency of oscillation of the wavy vortices were recorded.

OBSERVATIONS AND RESULTS

Observations on the vertical equipment were made for the most part through the edge of the gap. At low speeds of rotation, when $\mu = 0$, Couette flow was observed in the annulus. When the first critical point was reached, the alcohol-water drops separated into pairs of spiral vortices which counterrotated along the length of the cylinders. As the speed of the inner cylinder slowly increased, these vortices became stronger and their asymmetrical nature, as described by Appel (*10*), could be observed. As the speed increased further, the steady vortex system finally broke down in the following manner. At first the vortices waved in an axial direction without any apparent change in cross section. After a short time, every alternate vortex seemed to expand as the intermediate vortices contracted. This phenomenon is similar to that observed by Taylor (*1*). Finally, the drops diffused so that the boundaries between vortices could not be seen. As the speed increased

TABLE 1.

Isothermal criticality data $\mu = 0$, $Fg = 1.161$, $r_1 = 1.486$ in., $d = 0.269$ in.

ν (cs.)	n_{1c} (rev./min.)	n_{1w} (rev./min.)
13.84	57.3	65.3
13.84	57.1	65.3
22.02	87.8	103.4
18.80	74.8	86.0
26.94	100.2	124.0
6.05	24.5	28.8

Isothermal criticality data $\mu < 0$

ν (cs.)	n_2 (rev./min.)	n_{1c} (rev./min.)	n_{1w} (rev./min.)	μ_c	μ_w	P_w/P_c
15.1	10.7	63.2	70.6	−0.170	−0.152	1.140
15.0	14.0	60.7	68.9	−0.231	−0.203	1.190
14.9	16.4	61.6	69.0	−0.267	−0.237	1.194
14.5	28.4	65.3	71.0	−0.436	−0.400	1.167
14.7	35.7	68.9	74.3	−0.518	−0.481	1.156
16.6	45.4	81.6	85.2	−0.557	−0.533	1.097
14.1	47.1	75.0	76.9	−0.628	−0.613	1.026
13.9	55.1	79.9	81.2	−0.689	−0.678	1.024
15.2	55.6	83.3	84.5	−0.667	−0.658	1.021
14.7	57.2	81.7	82.7	−0.699	−0.692	1.011
22.0	26.8	93.0	103.4	−0.288	−0.259	1.136
19.0	54.2	96.3	102.1	−0.563	−0.530	1.086
21.0	55.4	100.0	112.1	−0.552	−0.496	1.195
21.3	76.0	112.2	116.8	−0.677	−0.658	1.070

* For a more complete description of the apparatus see Ho, C.Y., Ph. D. thesis, Rensselaer Polytechnic Institute, Troy, New York (1962), and Nardacci, J. L., Ph. D. thesis, Rensselaer Polytechnic Institute, Troy, New York (1962).

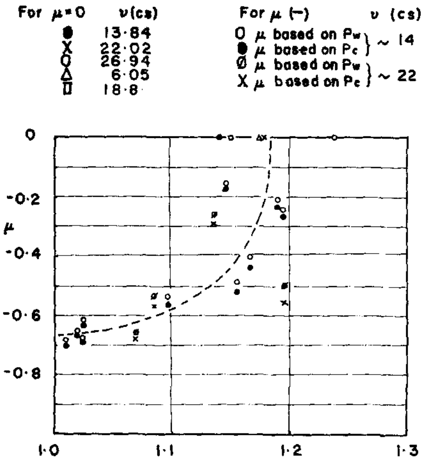

Fig. 3. Ratios of critical Taylor numbers for wavy vortices *Pw* to critical Taylor numbers for steady vortices *Pc* at different speed ratios μ; cylinders in counterrotation.

further, a periodic motion, which could not be accurately discerned by eye, would predominate. However, the frequency of these oscillations was observed and found to average about 0.85 to 0.90 oscillations per rotation for each oil considered.

When both cylinders were rotating, the observed phenomena were quite similar to those described above. At low speeds of rotation of the inner cylinder, the Couette pattern could be seen; the zero velocity streamline could easily be detected. At the first critical point, the steady vortex system appeared, and extended only across a portion of the annulus. In the region close to the outer cylinder, laminar Couette flow still seemed to predominate. As the inner cylinder speed further increased, the vortices broke down in the manner described above. However, in this case, as the speed of the outer cylinder was set at increasingly higher values, the revolutions per minute of the inner cylinder decreased between steady and wavy vortex formation. From the observed speeds of rotation for the critical points, the following results were obtained. In Figure 2 values of μ for steady vortex formation are plotted against μ for wavy vortex formation for each setting of the outer cylinder speed. (See Table 1 for values of μ.) The 45 deg. line on this plot represents values of equal μ for both steady and wavy vortex formation; that is, the wavy vortex system should form simultaneously upon formation of the steady vortex system. From the extrapolated curves through the data points, this phenomenon should occur at $\mu \simeq -0.73$.

To compare the data on a dimensionless base, values of the parameter P were calculated for the critical points; Pc was the value of P at the formation of the steady vortices and Pw the value for wavy vortex formation. Values of $\frac{Pw}{Pc}$ vs. μ were then plotted (Figure 3), and it was found that the simultaneous steady-wavy vortex system should form at approximately $\mu = -0.66$.

In all experiments where μ was between -0.70 and -0.75 only wavy vortices could be produced; steady nonwavy vortices were entirely absent.*

For $0 < \mu < 1$ the following observations were made. The outer cylinder speed was set at about 33 rev./min., and the speed of the inner cylinder increased from 0 rev./min. At about 115 rev./min. steady vortex formation was observed (corresponding to $\mu = 0.29$). These vortices remained steady as the speed of the inner cylinder was increased to its upper limit of about 300 rev./min.

With the outer cylinder at rest, the speed of the inner cylinder was increased until the wavy vortex pattern was firmly established. The speed of the outer cylinder was then increased, and gradually the wavy pattern broke down into a steady vortex system. As the speed of the outer cylinder was lowered, the opposite occurred; that is, the steady vortex system again gave way to the wavy system.

From the above observations, it is apparent that rotation of the outer cylinder in the same direction as the inner cylinder stabilizes the flow against the formation of wavy vortices. However, more work is needed in this area before any definite conclusions can be reached.

In the horizontal equipment, aluminum powder was used to visualize the flow. The aluminum powder-oil mixture was excellent for both visual and photographic observations.

With only one oil, it was found that the wavy vortices formed at a value of approximately $Pw/Pc = 1.21$ (the median between the lowest observed value of 1.18 and the highest of 1.24); their frequency upon formation was about 1 complete wave/rev.

The speeds at the critical points were recorded, and the flow pattern between the first and second critical points was carefully observed. For $1 < \frac{P}{Pc} < 1.05$, the steady vortices occupied square compartments (Figure 4*a*). As $\frac{P}{Pc}$ increased from 1.05 to 1.21, the vortices became nonsymmetrical and formed into distinct pairs (Figure 4*b*). These nonsymmetrical vortex pairs became stronger and more distinct as the second critical point was approached. At the second critical point these vortices, as in the vertical equipment, broke down by first beginning to wave in an axial direction and then alternating in cross section.

As the speed was further increased, the amplitude of the wavy vortices became smaller, but the frequency be-

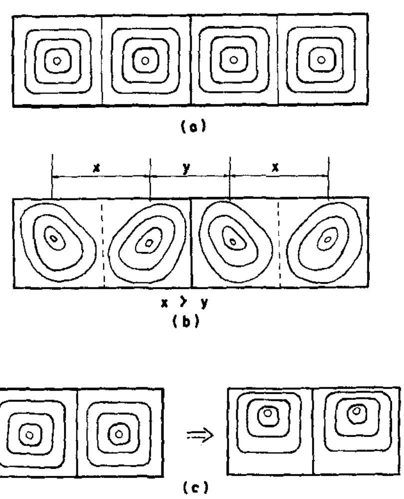

Fig. 4. (*a*) Vortex pattern for $1 < \frac{P}{Pc} <$ 1.05; (*b*) vortex pattern for $1.05 < \frac{P}{Pc} < 1.21$; (*c*) change of vortex pattern as $\frac{P}{Pc}$ changed from 1.05 to 7. All views are in radial sections. Outer cylinder stationary.

* J. W. Lewis (*11*) agreed qualitatively with the above observation. In a discussion of his experimental results for negative μ, he says, "It is found that, provided the speed is kept constant, the pulsating motion (refers to wavy vortices) persists indefinitely and with fixed amplitude. There appears to be no consistency in the speeds at which the pulsating type of motion sets in: sometimes the motion changed straightway at the critical point (refers to critical point for steady vortices) to the pulsating type."

came larger. At about $P/Pc = 7$, this combination of small amplitude and high frequency would cause the vortices once again to appear steady and to have clear boundary lines between each pair. Up to about $P/Pc = 15$, these low amplitude-high frequency vortices remained but seemed to become eccentric toward the outer cylinder as the speed was increased (Figure 4*c*).

At about $P/Pc = 15$, the boundaries between the vortices became less distinct. However, up to $P/Pc = 40$ (the highest value that could be attained) these boundaries, although very faint, could still be distinguished. This indicates that turbulence had not yet been reached. (Sketches of the above phenomenon are in Figure 5.)

DISCUSSION OF RESULTS

In the present study, it has been found that for $\mu = 0$ and $\frac{d}{r_1} \simeq \frac{1}{6}$, $Pw/Pc = 1.18$. Data of Coles (see Di Prima (*9*)) show that for $\mu = 0$ and $\frac{d}{r_1} = \frac{1}{8}$, at a speed about 29% higher than the first critical speed, a wave motion with four waves in the circumferential direction appeared.

These results agree qualitatively as both predict the formation of the wavy vortices at speeds slightly higher than the first critical speed. However, the reason for the quantitative discrepancy is not known although the difference in clearance ratios may be an important factor.

Since the above experiments were performed, several papers that deal theoretically with the formation of asymmetric vortices have been reviewed. Krueger and Di Prima (*12*) studied mathematically the stability of nonrotationally symmetric disturbances for $0 < \mu < 1$ for inviscid flow and found that the "growth rate of rotationally symmetric disturbances is greater than that of nonrotationally symmetric disturbances."

For $\mu < 0$, Bisshopp (*13*) studied mathematically the stability of steady flow for inviscid fluids between rotating cylinders. He found that the "growth rates of unstable modes which are very nearly symmetric are always less than the growth rate of the corresponding axisymmetric mode."

However, in a study of the stability characteristics of viscous Couette flow, Krueger (*14*) obtained results which indicate that for "μ greater than about -0.75 Couette flow is more stable to nonsymmetrical disturbances than to symmetrical disturbances. For μ less than about -0.75 the reverse is true."

The present experiments and the results of Krueger and Di Prima seem to agree well since both indicate that the steady vortex system is the more stable for $0 < \mu < 1$.

The case for $\mu < 0$ is more complex. Both Krueger's theory and the present extrapolated experimental data seem to indicate that for μ between -0.65 and -0.75, the nonsymmetric modes of disturbance become more unstable than the symmetric ones. Another possibility, however, was mentioned by Krueger. He postulated that the symmetric modes appear at the onset of instability for all $\mu < 0$, but below a certain value of μ these symmetric modes are very unstable and are readily superceded by nonsymmetric modes.

As previously mentioned, the steady vortices in the present experiments could not be observed below a value of $\mu = -0.7$ to -0.75, this agrees with the former of the above postulates. However, owing to the difficulties in obtaining small incremental changes in the speed of rotation with the transmissions used in this experiment, the above agreement should not be considered conclusive.

Nevertheless, from both theory and experiment, the nonsymmetric disturbances appear to be the dominant mode for $\mu = -0.7$ to -0.75.

As a check on the accuracy of the results, values of the critical Taylor number for $\mu = 0$ were calculated for the fluids used in the present experiments and were found to give a maximum deviation of about 5% from the theoretical Ta_c in all cases except one. For $\mu < 0$, experimentally measured values of Pc were compared with theoretically predicted values, and a maximum deviation of approximately 10% was found; however, most values showed only about a 7 or 8% deviation.

SUMMARY

The criteria for the formation of rotationally symmetric vortices for different modes of flow have been reviewed. In addition, the formation of nonrotationally symmetric or wavy vortices has been observed in both a horizontal and vertical apparatus. The value of the second critical point for $\mu = 0$ varies slightly for the two cases but averages about 20% higher than the first critical point. In both cases the frequency of the wavy vortices was about 1 wave/rev. (μ = ratio of revolutions per minute of outer to inner cylinder.)

For $\mu < 0$, the critical point for the formation of wavy vortices approached that of the steady vortices until at a value of μ from -0.66 to -0.73 the vortices would oscillate immediately upon their formation.

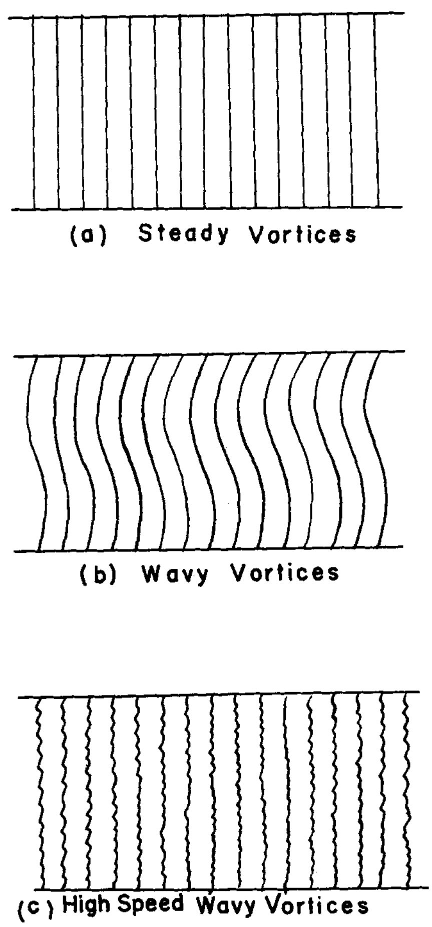

Fig. 5 (*a*) Vortex pattern for $1 < \frac{P}{Pc} < 1.21$. (*b*) vortex pattern for $1.21 < \frac{P}{Pc} < 7$; (*c*) vortex pattern for $\frac{P}{Pc} > 7$. All views are in elevation; that is, as they appear to external observer. Outer cylinder stationary.

The data for $0 < \mu < 1$ are inconclusive but seem to indicate that the rotation of the outer cylinder stabilizes the flow against the formation of wavy vortices.

Observations of the flow patterns between the first and second critical points (for $\mu = 0$) were also made, and the appearance of a transition region between steady square Taylor vortices and wavy vortices was noticed.

ACKNOWLEDGMENT

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NOTATION

d = gap width
Fg = correction factor for finite gap widths
n = speed of rotation of cylinders (rev./min.)
P = stability parameter based on δ
r = radius
r_0 = radius of zero velocity streamline
Ta = Taylor number
$\overline{V}$ = average velocity of flow in potentially unstable region of annulus
Z = symbol used in defining Fg
v = velocity
δ = potentially unstable region of gap based on Couette flow equation
μ = ratio of cylinder speeds (ω_1/ω_2); negative for rotation in opposite directions
ν = kinematic viscosity
ω = speed of rotation of cylinders (radian/sec.)

Subscripts

1 = inner cylinder
2 = outer cylinder
c = critical point for steady vortices
w = critical point for wavy vortices
m = average value, for example $r_m = \frac{r_1 + r_2}{2}$

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Experimental Study of Laminar Flow Heat Transfer with Prescribed Wall Heat Flux

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Laminar-flow heat transfer in a cylindrical tube was studied experimentally in order to test a recent analytical method of predicting temperature profiles (1). The experimental equipment was designed to measure wall temperatures to better than 0.1°C. and to utilize small radial heat fluxes so as to minimize effects of natural convection.

A series of experiments was carried out on five different types of prescribed wall heat flux with water flowing in a vertical tube. The measured wall temperatures in these experiments compared well with those predicted by the theoretical solution. Small deviations which did exist were explained in terms of experimental errors.

The agreement found indicates that the analytical method may be safely used in the study of experimental data in such heat transfer and analogous mass transfer problems.

Ever since the classical work of Graetz in 1883 (2) there has been considerable interest in the problem of laminar-flow heat transfer. Many analytical solutions are available for the prediction of temperature in laminar-flow fields for various geometries and boundary conditions and with certain simplifying assumptions regarding fluid properties. Recently solutions have appeared for the case in which the flowing fluid is heated in a tube in which the wall heat flux varies in the direction of flow. Solutions of this type are useful not only in the heat transfer con-

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